Name of College – S.S. College J.Bad (1)
Deptt – Mathematics
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TOPIC – Problem based on Inverse Hyperbolic Functions

B.Sc. I (Hons)

Time – 10 A.M to 11.30 A.M

Date – 12-04-2021

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* Important formulae.

(i) If $z = \sinh^{-1} w \Rightarrow z = n\pi i + (-1)^n \log[w + \sqrt{w^2+1}]$ (General value)

$$z = \log[w + \sqrt{w^2+1}] \text{ (Principal value)}$$

(ii) If $z = \cosh^{-1} w \Rightarrow z = 2n\pi i \pm \log[w \pm \sqrt{w^2-1}]$ [General value]

$$z = \log[w + \sqrt{w^2-1}] \text{ [Principal value]}$$

(iii) If $z = \tanh^{-1} w \Rightarrow z = n\pi i + \frac{1}{2}\log\frac{1+w}{1-w}$ (General value)

$$= \frac{1}{2}\log\frac{1+w}{1-w} \text{ (Principal value)}$$

2.* Relation between inverse hyperbolic functions and inverse circular functions

$\sinh^{-1} w = -i\sin^{-1}(iw)$
$\cosh^{-1} w = -i\cos^{-1}(w)$
$\tanh^{-1} w = -i\tan^{-1}(iw)$

Problem :->

Prove that $\sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \tanh^{-1}x$

Solution :-> let $\tanh^{-1}x = y$

$\Rightarrow \cancel{\tanh y} = \cancel{\tanh^{-1}x}$ x

$\Rightarrow x = \tanh y$

Since $\operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$

$\Rightarrow \operatorname{sech} y = \sqrt{1-x^2}$

$\Rightarrow \cosh y = \frac{1}{\sqrt{1-x^2}}$

Now $\because \cosh^2 y - \sinh^2 y = 1$

$\Rightarrow \sinh^2 y = \cosh^2 y - 1$

$= \frac{1}{1-x^2} - 1$

$= \frac{1 - 1 + x^2}{1-x^2}$

$= \frac{x^2}{1-x^2}$

$\therefore \sinh y = \frac{x}{\sqrt{1-x^2}}$

$\therefore y = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

$\Rightarrow \tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

Similarly Conversion of One inverse Hyperbolic Function Can be done into other inverse hyperbolic function too.

Prove that
$$\tan^{-1}\left(i\,\frac{x-a}{x+a}\right) = -\frac{i}{2}\log\frac{a}{x}$$

Solution :→ here $\tan^{-1}\left(i\,\frac{x-a}{x+a}\right)$

$$= i\tanh^{-1}\left[\frac{x-a}{x+a}\right]$$

Since. $\tanh^{-1}x = -i\tan^{-1}(ix)$

$$= i\,\frac{1}{2}\log\frac{1+\frac{x-a}{x+a}}{1-\frac{x-a}{x+a}}$$

Since $z = \tanh^{-1}w$
$$= \frac{1}{2}\log\frac{1+w}{1-w}.$$

$$= i\times\frac{1}{2}\log\frac{x+a+x-a}{x+a-x+a}$$

$$= i\times\frac{1}{2}\log\frac{x}{a}$$

$$= i\times\frac{1}{2}\log\left(\frac{a}{x}\right)^{-1}$$

$$= -\frac{i}{2}\log\frac{a}{x}$$

Proved

3. Separate $\sin^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts :→

Solution :→ Let $\sin^{-1}(\cos\theta + i\sin\theta) = x+iy$

$$\Rightarrow \cos\theta + i\sin\theta = \sin(x+iy)$$

$$\Rightarrow \cos\theta + i\sin\theta = \sin x\cos iy + \cos x\sin iy$$

$$= \sin x\cosh y + i\cos x\sinh y$$

Equating real and imaginary parts

$\sin x\cosh y = \cos\theta$ ——— (I)

$\cos x\sinh y = \sin\theta$ ——— (II)

$\because \cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = 1$

$\Rightarrow \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y = 1$

$\Rightarrow \sin^2 x + \sin^2 x \sinh^2 y + \sinh^2 y - \sin^2 x \sinh^2 y = 1$

$\Rightarrow \sinh^2 y = 1 - \sin^2 x = \cos^2 x$

$\Rightarrow \sinh y = \cos x$

From (2) we have

$\cos^2 x = \sin\theta$ Only +ve sing of $\sin\theta$ is considered

Since $-\pi/2 < x < \pi/2$

$\cos x = \sqrt{\sin\theta}$

$\Rightarrow x = \cos^{-1}\sqrt{\sin\theta}$

From equation (2) $\sinh y = \sqrt{\sin\theta}$.

~~$\cos^2 x = \sin\theta$~~

$\Rightarrow y = \sinh^{-1}\sqrt{\sin\theta}$

$= \log\left[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right]$

Thus $\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}\sqrt{\sin\theta} + i\log\left[\sqrt{\sin\theta} + \sqrt{1+\sin\theta}\right]$

* Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ in to real and imaginary parts —

[Do your self]